



# Particles with curvature and torsion in three-dimensional pseudo-Riemannian space forms

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## Abstract

We consider the motion of relativistic particles described by an action which is a function of the curvature and torsion of the particle path. The Euler–Lagrange equations and the dynamical constants of the motion are expressed in a simple way in terms of a suitable coordinate system. The moduli spaces of solutions in a three-dimensional pseudo-Riemannian space form are completely exhibited.

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## 1. Introduction

This paper deals with the motion of relativistic particles described by an action which is a function of the curvature and torsion of the particle path. The model of a particle with torsion was investigated in (2+1)-Minkowskian space [1]. It was shown that, at classical level, the squared mass of the system is bounded from above and that, besides the massive solutions of the equations

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of motion, the model must also have massless and tachyonic solutions. A relativistic model of the anyon, describing the states of the particle with torsion  $\tau$  with the maximum value of the mass, was constructed in ref. [2]. In ref. [3], the classical equations of motion of the model whose Lagrangian function is  $f = -m + \alpha\tau$  are obtained. This model of relativistic particle with torsion (whose action appears in the Bose–Fermi transmutation mechanism) is also studied in ref. [4], where it is canonically quantized in the  $(2 + 1)$ -Minkowskian and 3-Euclidean spaces. The solutions of the equations of motion in the massive, massless and tachyonic sectors are found by using Hamiltonian formalism. In ref. [5], the author reconsiders the simplest models describing spinning particles with rigidity, both massive and massless, and describes the moduli spaces of solutions in  $(2+1)$ -backgrounds with constant curvature.

In  $(3 + 1)$ -dimensions there are also some geometrical models of relativistic particles. It seems interesting to investigate these models and to establish which of them have a maximal symmetry [6]. For instance,  $f = a + bk$ ,  $a \neq 0$ , describes a massive relativistic boson [7];  $f = ck$  models a massless particle with an arbitrary (both integer and half-integer) helicity [8]. In ref. [9], the author studied the consequences of coupling of the highest curvature to the Lagrangian of a massive spinless particle. More recently, in ref. [10], it is considered a relativistic particle whose dynamics is determined by an action depending on the torsion  $\tau$ . The Euler–Lagrange equations are obtained but unfortunately, as the authors pointed out, these higher order differential equations do not appear to be tractable in general. In ref. [11], we also consider mechanical systems linearly depending on the curvature and the torsion of the particle path and obtain the moduli space of solutions in four-dimensional background spaces.

The main purpose of this paper is providing the moduli space of solutions of mechanical systems, in three-dimensional pseudo-Riemannian space forms, whose Lagrangian is an arbitrary function on the curvature and torsion of the particle path. In Sections 2 and 3, we present the model, whose action is given by

$$\mathcal{L}(\gamma) = \int_{\gamma} f(k, \tau) ds,$$

where  $f$  is a real arbitrary function. By using Killing vector fields along curves as a key tool, we obtain and solve the motion equations for these Lagrangians. In Section 4, we integrate the Frenet equations finding out the critical curves, which are critical points of the Lagrangian, in terms of a suitable coordinate system. We point out that a similar study for flat spaces has been realized in refs. [10,12], where it is shown that the trajectories corresponding to a Lagrangian with a linear dependence on  $k$  are determined by a quadrature in  $\tau$ . With the aim of getting a nearly self-contained paper, in Appendix A, we include an appendix about the Lie algebras  $\mathfrak{o}(4, \nu)$ ,  $\nu = 0, 1, 2$ .

## 2. The model and the motion equations

Let  $M_{\nu}^3(C)$  be a three-dimensional pseudo-Riemannian space form of curvature  $C$  and index  $\nu$ . Let  $\gamma : I \rightarrow M_{\nu}^3(C)$  be an immersed curve with speed  $v(t) = |\gamma'(t)|$ , curvature  $k$ , torsion  $\tau$  and Frenet frame  $\{T, N, B\}$ . The Frenet equations are written down as follows

$$\begin{cases} \nabla_T T = \varepsilon_2 k N, \\ \nabla_T N = -\varepsilon_1 k T + \varepsilon_3 \tau B, \\ \nabla_T B = -\varepsilon_2 \tau N, \end{cases}$$

where  $\varepsilon_1 = \langle T, T \rangle$ ,  $\varepsilon_2 = \langle N, N \rangle$  and  $\varepsilon_3 = \langle B, B \rangle$ . Let

$$\mathcal{L}(\gamma) = \int_{\gamma} f(k, \tau) ds \tag{1}$$

be the action for any real function  $f$  defined on an open set of  $\mathbb{R}^2$ . Let  $\Gamma = \Gamma(t, r) : [0, L] \times (-\delta, \delta) \rightarrow M$  be a variation of a curve  $\gamma : [0, L] \rightarrow M$  with  $\Gamma(t, 0) = \gamma(t)$ . Associated with  $\Gamma$ , we consider the variation vector field  $W = W(t) = \frac{\partial \Gamma}{\partial r}(t, 0)$  along  $\gamma(t)$ . We also write  $V = V(t, r) = \frac{\partial \Gamma}{\partial t}(t, r)$ ,  $W = W(t, r)$ ,  $v = v(t, r)$ ,  $T = T(t, r)$ ,  $N = N(t, r)$ ,  $B = B(t, r)$ , etc., with the obvious meanings. Let  $s$  denote the arclength parameter, and let  $V(s, r)$ ,  $W(s, r)$ , etc., be the corresponding reparametrizations. To obtain the first variation equation we introduce general formulas for the variations of  $v$ ,  $\kappa$  and  $\tau$  along  $\gamma$  in the direction of  $W$ . Then from the Frenet equations we obtain

$$\begin{aligned} W(v) &= \varepsilon_1 v \langle \nabla_T W, T \rangle, \quad W(k) = \langle \nabla_T^2 W, N \rangle - 2\varepsilon_1 k \langle \nabla_T W, T \rangle + \varepsilon_1 C \langle W, N \rangle, \\ W(\tau) &= \varepsilon_1 k \langle \nabla_T W, B \rangle - \varepsilon_1 \tau \langle \nabla_T W, T \rangle + T \left( \frac{\varepsilon_2}{k} (\langle \nabla_T^2 W, B \rangle + \varepsilon_1 C \langle W, B \rangle) \right). \end{aligned}$$

Throughout this paper, the trivial case of geodesics ( $k = 0$ ) will be excluded. Then, by using standard arguments involving the above formulas and integration by parts, the first variation of  $\mathcal{L}(\gamma)$  along  $\gamma$  in the direction of  $W$  is given by

$$\mathcal{L}'(0) = [\mathcal{B}(\gamma, W)]_0^L - \int_0^L \left\langle \nabla_T P - \varepsilon_1 C f_k N + \varepsilon_1 \varepsilon_2 C \frac{f'_\tau}{k} B, W \right\rangle ds, \tag{2}$$

where the vector  $P$  is given by

$$P = \varepsilon_1 (f - (2kf_k + \tau f_\tau))T + \varepsilon_1 k f_\tau B - \nabla_T (f_k N) + \varepsilon_2 \nabla_T \left( \frac{f'_\tau}{k} B \right)$$

and the boundary term is

$$\mathcal{B}(\gamma, W) = \left\langle \nabla_T^2 W, \varepsilon_2 \frac{f'_\tau}{k} B \right\rangle + \left\langle \nabla_T W, f_k N - \frac{\varepsilon_2}{k} f'_\tau B \right\rangle + \left\langle W, P + \frac{\varepsilon_1 \varepsilon_2 C f'_\tau}{k} B \right\rangle.$$

Observe that, we have used  $f_k$  and  $f_\tau$  to denote the partial derivatives of  $f$  with respect to  $k$  and  $\tau$ , respectively. On the other hand, we restrict ourselves to variations with fixed endpoints having the same Frenet frames on them. Then  $[\mathcal{B}(\gamma, W)]_0^L = 0$ , so that the critical curves are characterized by the vanishing of the Euler–Lagrange operator  $\mathcal{E}$

$$\mathcal{E} := - \left( \nabla_T P - \varepsilon_1 C f_k N + \varepsilon_1 \varepsilon_2 C \frac{f'_\tau}{k} B \right) = 0. \tag{3}$$

It is a straightforward computation to show that Eq. (3) is equivalent to the Euler–Lagrange equations

$$-\varepsilon_1 \varepsilon_2 k f - \varepsilon_2 (\varepsilon_3 \tau^2 - \varepsilon_1 k^2) f_k + 2\varepsilon_1 \varepsilon_2 k \tau f_\tau + f''_k + \left( \tau \frac{f'_\tau}{k} \right)' + \tau \left( \frac{f'_\tau}{k} \right)' + \varepsilon_1 C f_k = 0, \tag{4}$$

$$\varepsilon_3 \tau f'_k + \varepsilon_3 \frac{\tau^2}{k} f'_\tau + \varepsilon_3 (\tau f_k)' - \varepsilon_1 (k f_\tau)' - \varepsilon_2 \left( \frac{f'_\tau}{k} \right)'' - \varepsilon_1 \varepsilon_2 C \frac{f'_\tau}{k} = 0. \tag{5}$$

### 3. Solving the motion equations

We are going to integrate the motion equations. To do that, we will use the ideas involved in Noether’s theorem in order to get invariants which provide suitable integral equations. We will first obtain two very useful Killing vector fields.

**Proposition 1.** *The critical curves of the Lagrangian (1) admit two Killing vector fields  $P$  and  $J$  given by*

$$P = \varepsilon_1(f - (kf_k + \tau f_\tau))T - \left(f'_k + \frac{\tau}{k}f'_\tau\right)N + \left(-\varepsilon_3\tau f_k + \varepsilon_1kf_\tau + \varepsilon_2\left(\frac{f'_\tau}{k}\right)'\right)B, \tag{6}$$

$$J = -\varepsilon_1f_\tau T - \frac{f'_\tau}{k}N - \varepsilon_3f_k B, \tag{7}$$

satisfying that

- (i)  $\mathcal{E} = -(\nabla_T P + \varepsilon C J \wedge T)$
- (ii)  $\nabla_T J = -P \wedge T$

where  $\varepsilon = \varepsilon_1\varepsilon_2\varepsilon_3$ .

**Proof.** In a flat space, the Euler–Lagrange equations yield  $P$  is a constant vector field along the curve. On the other hand, from the Frenet equations, a direct computation allows us to write  $P$  as

$$P = \varepsilon_1(f - (kf_k + \tau f_\tau))T - \left(f'_k + \frac{\tau}{k}f'_\tau\right)N + \left(-\varepsilon_3\tau f_k + \varepsilon_1kf_\tau + \varepsilon_2\left(\frac{f'_\tau}{k}\right)'\right)B.$$

Let  $Z$  be a constant vector field and choose a rotational vector field of the form  $W = \gamma \wedge Z$  as variational vector field. Then, we find that the curves of the variation have the same curvature and torsion functions as the starting curve, so that  $\mathcal{L}'(0) = 0$ . Now, from (2) we get

$$\mathcal{L}'(0) = [\mathcal{B}(\gamma, \gamma \wedge Z)]_0^L + \int_0^L \langle \mathcal{E}, \gamma \wedge Z \rangle ds,$$

so that, as the Euler operator vanishes on critical curves, we have that

$$[\mathcal{B}(\gamma, \gamma \wedge Z)]_0^L = 0.$$

Finally, as the same reasoning holds for any real in the interval  $(0, L)$ , we find that  $\mathcal{B}(\gamma, \gamma \wedge Z)$  is constant along critical curves. As  $Z$  was any constant vector field, we conclude that

$$\left\langle \nabla_T T \wedge Z, \varepsilon_2 \frac{f'_\tau}{k} B \right\rangle + \left\langle T \wedge Z, f_k N - \varepsilon_2 \frac{f'_\tau}{k} B \right\rangle + \langle \gamma \wedge Z, P \rangle = \text{constant}.$$

Operating here, we obtain  $\langle J - \gamma \wedge P, Z \rangle = \text{constant}$ , where

$$J = -\varepsilon_1f_\tau T - \frac{f'_\tau}{k}N - \varepsilon_3f_k B.$$

Therefore,  $V = J - \gamma \wedge P$  is a constant vector field, which means that  $J$  is a translation followed by a rotation, and so it is a Killing vector field.

Then, we have shown that  $P$  and  $J$  are restrictions, along critical curves, of Killing fields in flat spaces. Following Langer and Singer (see ref. [13]), we define a Killing vector field along a

curve as a vector field  $W$  such that  $W(v) = W(k) = W(\tau) = 0$ . It is easy to show that, in space forms, a Killing vector field along a curve is the restriction of a Killing vector field. Then, from (i) and (ii), a straightforward computation shows that  $P$  and  $J$  are also Killing vector fields when  $C \neq 0$ .  $\square$

The power of Killing vector fields as a tool is pointed out in the following result.

**Theorem 2. (Integral equations)** *The critical curves of the Lagrangian (1) satisfy the integral equations*

$$\begin{cases} \langle P, P \rangle + \varepsilon C \langle J, J \rangle = d, \\ \langle P, J \rangle = e, \end{cases} \tag{8}$$

for suitable constants  $d$  and  $e$ . They are written, in terms of  $f$ , as

$$\begin{aligned} &\varepsilon_1(f - (kf_k + \tau f_\tau))^2 + \varepsilon_2 \left( f'_k + \frac{\tau}{k} f'_\tau \right)^2 + \varepsilon_3 \left( -\varepsilon_3 \tau f_k + \varepsilon_1 k f_\tau + \varepsilon_2 \left( \frac{f'_\tau}{k} \right)' \right)^2 \\ &+ \varepsilon C \left( \varepsilon_1 f_\tau^2 + \varepsilon_2 \left( \frac{f'_\tau}{k} \right)^2 + \varepsilon_3 f_k^2 \right) = d, \\ &-\varepsilon_1 f_\tau (f - (kf_k + \tau f_\tau)) + \varepsilon_2 \frac{f'_\tau}{k} \left( f'_k + \frac{\tau}{k} f'_\tau \right) \\ &- f_k \left( -\varepsilon_3 \tau f_k + \varepsilon_1 k f_\tau + \varepsilon_2 \left( \frac{f'_\tau}{k} \right)' \right) = e. \end{aligned}$$

Furthermore, these equations are equivalent to the Euler–Lagrange ones provided that  $\langle J, J \rangle$  is not constant.

**Proof.** From Proposition 1, we get

$$\begin{aligned} \langle \mathcal{E}, P \rangle &= -\frac{1}{2} T(\langle P, P \rangle + \varepsilon C \langle J, J \rangle), \\ \langle \mathcal{E}, J \rangle &= -T \langle P, J \rangle, \end{aligned}$$

from which we deduce (8). Now, we see that the equivalence between (8) and the Euler–Lagrange equations occurs when  $\langle P, J \wedge T \rangle$  does not vanish. But this condition means that the systems

$$\begin{cases} \langle \mathcal{E}, P \rangle = 0 \\ \langle \mathcal{E}, J \rangle = 0 \end{cases} \quad \text{and} \quad \begin{cases} \langle \mathcal{E}, N \rangle = 0 \\ \langle \mathcal{E}, B \rangle = 0 \end{cases}$$

are equivalent. As  $\nabla_T J = -P \wedge T$ , we have that  $\langle P, J \wedge T \rangle = \frac{1}{2} T \langle J, J \rangle$ , which proves the equivalence of the systems provided  $\langle J, J \rangle$  is not constant.  $\square$

**Remark.** The constants appearing in Theorem 2 can be interpreted in terms of the mass  $M$  and the spin  $S$  of the particle. Indeed, generalizing Plyushchay’s model the relationships are given by

$$\langle P, P \rangle + \varepsilon C \langle J, J \rangle = \varepsilon_1 M^2, \quad \langle P, J \rangle = MS.$$

When  $C = 0$ ,  $P$  and  $J$  can be interpreted as the linear and the angular, respectively, momenta of the particle.

3.1. Integrating the Euler–Lagrange equations when  $\langle J, J \rangle$  is constant

It is easy to see, from Proposition 1, that  $\langle P, J \wedge T \rangle = 0$ , provided  $\langle J, J \rangle$  is constant. Then  $P$ ,  $J$  and  $T$  are linearly dependent. Moreover, when  $\langle J, T \rangle$  is constant,  $f_k$  and  $f_\tau$  also are constant, because

$$f_\tau = -\langle J, T \rangle \quad \text{and} \quad f_k^2 = \langle J, B \rangle^2 = \varepsilon_3(\langle J, J \rangle - \varepsilon_1 \langle J, T \rangle^2).$$

Then the critical curves of the Lagrangian density  $f(k, \tau) = f_k k + f_\tau \tau + m$  are well known and they are generalized helices (see refs. [5] and [11]).

Then, we can assume that  $\langle J, T \rangle$  is not constant. Furthermore,  $J$  and  $T$  are not collinear, otherwise  $\langle J, T \rangle^2 = \varepsilon_1 \langle J, J \rangle$ . Therefore, the frame  $\{T, J, J \wedge T\}$  allows us to get the following system of equations, which is equivalent to the Euler–Lagrange one,

$$\langle \nabla_T P + \varepsilon C J \wedge T, J \rangle = 0, \quad \langle \nabla_T P + \varepsilon C J \wedge T, J \wedge T \rangle = 0, \quad \langle J, J \rangle = \text{const.} \quad (9)$$

As

$$\langle \nabla_T P, J \rangle = T \langle P, J \rangle - \langle P, \nabla_T J \rangle = T \langle P, J \rangle,$$

from Proposition 1 the first equation of (9) writes down as  $\langle P, J \rangle = \text{const.}$

From the third equation, we deduce that  $P = \alpha T + \beta J$ . Therefore, when  $\beta \neq 0$ , the first equation reduces to  $\langle P, P \rangle = \text{const.}$ , because

$$\langle \nabla_T P, J \rangle = \left\langle \nabla_T P, \frac{1}{\beta}(P - \alpha T) \right\rangle = \frac{1}{\beta} \langle \nabla_T P, P \rangle,$$

and  $\langle \nabla_T P, T \rangle = 0$ . When  $\beta = 0$ , we see that  $\alpha$  should be constant. Summarizing, to solve the second equation of (9), we can suppose that  $\langle P, P \rangle$ ,  $\langle P, J \rangle$  and  $\langle J, J \rangle$  are all constant.

On the other hand, we compute

$$\langle \nabla_T P, J \wedge T \rangle + \varepsilon C \langle J \wedge T, J \wedge T \rangle = -|P \wedge T|^2 + \varepsilon C |J \wedge T|^2 - \varepsilon_2 k \langle P, J \wedge N \rangle,$$

where we have used that  $\langle P, J \wedge T \rangle = 0$  and  $\nabla_T J = -P \wedge T$ . Moreover, we can write

$$-|P \wedge T|^2 + \varepsilon C |J \wedge T|^2 = -\varepsilon_2 \varepsilon_3 \alpha k \langle J, B \rangle.$$

As  $P \wedge J = \alpha T \wedge J$ , we get

$$\alpha = \frac{\langle P \wedge J, T \wedge J \rangle}{|J \wedge T|^2} \quad \text{and} \quad \beta = \frac{\langle P, N \rangle}{\langle J, N \rangle} = k \frac{f'_k}{f'_\tau} + \tau$$

which are related by

$$\alpha = \varepsilon_1 (\langle P, T \rangle - \beta \langle J, T \rangle) = \varepsilon_1 \left( f - k f_k + k \frac{f'_k f_\tau}{f'_\tau} \right). \quad (10)$$

Note that  $\langle J, N \rangle \neq 0$ , because  $\langle J, T \rangle$  is not constant.

Now, we obtain a system, equivalent to (9), in terms of  $f$  and its partial derivatives:

$$\frac{f'_k}{f'_\tau} = \frac{\langle P, J \rangle - \tau \langle J, J \rangle + \varepsilon_1 (f - k f_k) f_\tau}{\varepsilon_1 \varepsilon k |J \wedge T|^2}, \quad (11)$$

$$\begin{aligned} (c_1 - \varepsilon C f_\tau^2 + (f - (k f_k + \tau f_\tau))^2) (\langle J, J \rangle \varepsilon_1 - f_\tau^2) \\ = \varepsilon_1 k f_k ((f - (k f_k + \tau f_\tau)) \langle J, J \rangle + f_\tau \langle P, J \rangle), \end{aligned} \quad (12)$$

$$\langle J, J \rangle = \varepsilon_1 f_\tau^2 + \varepsilon_2 \left( \frac{f'_\tau}{k} \right)^2 + \varepsilon_3 f_k^2, \tag{13}$$

where  $c_1 = \varepsilon_1(\varepsilon C \langle J, J \rangle - \langle P, P \rangle)$ .

Eq. (11) lead us to the relation

$$\tau'(Af_{\tau\tau} - f_{k\tau}) = k'(f_{kk} - Af_{\tau k}), \tag{14}$$

where

$$A(k, \tau) = \frac{\langle P, J \rangle - \tau \langle J, J \rangle + \varepsilon_1(f - kf_k)f_\tau}{\varepsilon_1 \varepsilon k |J \wedge T|^2}.$$

Furthermore, by assuming that  $f_{kk}f_{\tau\tau} - f_{k\tau}^2 \neq 0$ , it is easy to see that  $Af_{\tau\tau} - f_{k\tau}$  and  $f_{kk} - Af_{\tau k}$  do not vanish simultaneously. Let us choose the first one. Then, from the Eqs. (13) and (14) we get

$$(k')^2 = \varepsilon_2 \frac{k^2(Af_{\tau\tau} - f_{k\tau})^2}{(f_{kk}f_{\tau\tau} - f_{k\tau}^2)^2} (\langle J, J \rangle - \varepsilon_1 f_\tau^2 - \varepsilon_3 f_k^2).$$

Now, Eq. (12) is of the form  $F(k, \tau) = 0$ , so that when  $\frac{\partial F}{\partial \tau} \neq 0$ , we find  $\tau$  as a function of  $k$  and solve  $k$  by quadratures.

If  $f_{kk} - Af_{\tau k} \neq 0$ , we can proceed similarly.

**Remark.** Whether  $f(k, \tau)$  only depends on  $k$  or  $\tau$ , a straightforward computation shows that the only critical curves are generalized helices.

### 3.1.1. $P$ and $J$ are collinear

An interesting situation where  $\langle J, J \rangle$  is constant occurs when  $P = \beta J$ . Here, it is included the case where the critical curves are generalized helices but not classical helices, as it can be easily checked. We observe that, when  $J$  is zero, the solutions are geodesics. In this case, we cannot integrate the Frenet equations, but we can do the Euler–Lagrange ones. As  $\nabla_T J = -P \wedge T$ , we deduce that

$$\mathcal{E} = -\beta' J + (\beta^2 - \varepsilon C) J \wedge T = 0,$$

and

$$-\beta' \langle J, T \rangle = 0, \quad (\beta^2 - \varepsilon C) |J \wedge T|^2 = 0.$$

It is not difficult to see that this happens only when one the following cases holds:

- (i)  $J = 0$ ,
- (ii)  $\langle J, T \rangle$  and  $\beta$  are both constant and  $J = \varepsilon_1 \langle J, T \rangle T$ ,
- (iii)  $\beta^2 = \varepsilon C$ .

The solutions are geodesics in the two first cases, because they only occur when  $f_k$  and  $f_\tau$  both vanish. In the third case, the Euler–Lagrange equations are trivially satisfied. Therefore, to get the critical curves it is enough to look for curves satisfying the system

$$\langle P, T \rangle = \beta \langle J, T \rangle, \quad \langle P, N \rangle = \beta \langle J, N \rangle, \quad \langle P, B \rangle = \beta \langle J, B \rangle.$$

Moreover,  $\langle P, N \rangle = \beta \langle J, N \rangle$  can be deduced by taking covariant differentiation in  $\langle P, T \rangle = \beta \langle J, T \rangle$  relative to  $T$ , because  $P$  and  $J$  are Killing vector fields. Therefore, we have to solve the system

$$\begin{cases} f - kf_k = (\tau - \beta)f_\tau, \\ \varepsilon_1 kf_\tau + \varepsilon_2 \left(\frac{f'_\tau}{k}\right)' = \varepsilon_3 f_k(\tau - \beta). \end{cases}$$

By assuming that  $f_{kk}f_{\tau\tau} - f_{k\tau}^2 \neq 0$ , we can obtain  $k$  as a function of  $\tau$  or viceversa.

Assume that  $k$  is a function  $k = h(\tau)$ . Then

$$f'_\tau = k'f_{\tau k} + \tau'f_{\tau\tau} = \tau'(h_\tau f_{\tau k} + f_{\tau\tau}),$$

so that

$$\varepsilon_1 h f_\tau + \varepsilon_2 \left(\frac{\tau'(h_\tau f_{\tau k} + f_{\tau\tau})}{h}\right)' = \varepsilon_3 f_k(\tau - \beta).$$

This equation can be written as

$$a_1(\tau)(\tau')^2 + a_2(\tau)\tau'' = a_3(\tau).$$

This is an ordinary differential equation which can be transformed into a linear one by doing  $\tau' = w(\tau)$  and  $y(\tau) = w(\tau)^2$ . For instance, doing the calculations when  $f(k, \tau) = k^2 + \tau^2$ , we get

$$\tau' = \sqrt{\varepsilon_2(2\tau\beta - \tau^2)(\varepsilon_3(\tau^2 - 2\tau\beta) - \varepsilon_1\tau^2 + C_1)},$$

where  $C_1 \in \mathbb{R}$  and  $k^2 = 2\tau\beta - \tau^2$ . Therefore, we have found critical curves which are not generalized helices.

### 3.2. Integrating the Euler–Lagrange equations when $\langle J, J \rangle$ is not constant: two interesting cases

Setting  $Q = P - \tau J$ , the integrals of the Euler–Lagrange equations can be written as

$$\begin{cases} \langle Q, Q \rangle + \tau \langle Q, J \rangle + \varepsilon C \langle J, J \rangle = d - \tau e, \\ \langle Q, J \rangle + \tau \langle J, J \rangle = e. \end{cases} \tag{15}$$

Although these equations cannot be, in general, integrated, some particular cases such as  $f(k, \tau) = g(k) + a\tau$  or  $f(k, \tau) = g(\tau)$  deserve our attention.

**(3.1.a)**  $f(k, \tau) = g(k) + a\tau$

From the second equation of (15), we find out

$$\tau = \frac{e + \varepsilon_1 a g}{\varepsilon_3 g_k^2},$$

which we bring to the first equation of (15) to get  $k$  by quadratures

$$(k')^2 = \frac{g_k^2(d - \varepsilon_1(g - kg_k)^2 - \varepsilon_3 a^2 k^2 - \varepsilon C(\varepsilon_1 a^2 + \varepsilon_3 g_k^2)) - \varepsilon_3(e + \varepsilon_1 a g)(e + \varepsilon_1 a g - 2\varepsilon_1 a k g_k)}{\varepsilon_2 g_k^2 g_{kk}^2}.$$

Note that  $g_k^2 g_{kk}^2 \neq 0$ , because we had assumed that  $\langle J, J \rangle$  is not constant.



**(3.1.b)**  $f(k, \tau) = g(\tau)$ . For a physical meaning of a model with torsion see ref. [4] and references therein.

Now, we first observe that from the first Euler–Lagrange equation we deduce that

$$\varepsilon_1 k f_\tau + \varepsilon_2 \left( \frac{f'_\tau}{k} \right)' = \frac{\varepsilon_2}{2\tau} \left( \varepsilon_1 \varepsilon_2 k f - \frac{\tau' f'_\tau}{k} \right).$$

From here,  $Q$  can be written as

$$Q = \varepsilon_1 g T + \frac{\varepsilon_2}{2\tau} \left( \varepsilon_1 \varepsilon_2 k g - \tau' \frac{g'_\tau}{k} \right) B.$$

Then, from the first equation of (15), we get

$$(\tau')^2 = \frac{k^2(e + \varepsilon_1 g_\tau(g - \tau g_\tau))}{\varepsilon_2 \tau g_{\tau\tau}^2}. \tag{16}$$

We also have that  $\tau g_{\tau\tau}^2 \neq 0$ , because  $\langle J, J \rangle$  is not constant. A straightforward computation involving (16) gives

$$\varepsilon_1 g_\tau^2 + \varepsilon_2 \left( \frac{g'_\tau}{k} \right)^2 = \frac{e + \varepsilon_1 g_\tau}{\tau}.$$

Then, we can obtain  $k$  as a function of  $\tau$  and therefore (16) can be rewritten as

$$k^2 = \frac{4\varepsilon_3 \tau^3 g_{\tau\tau}^2 (\tau(d - \tau e - \varepsilon_1 g^2 + \varepsilon_1 \tau g g_\tau) - \varepsilon C(\varepsilon_1 g_\tau g + e))}{(\tau(g g_{\tau\tau} + g_\tau^2) - \varepsilon_1 e - g g_\tau)^2}.$$

Finally, we get

$$(\tau')^2 = \frac{4\varepsilon_2 \varepsilon_3 \tau^2 (\tau(d - \tau e - \varepsilon_1 g^2 + \varepsilon_1 \tau g g_\tau) - \varepsilon C(\varepsilon_1 g_\tau g + e)) (\varepsilon_1 g_\tau (g - \tau g_\tau) + e)}{(\tau(g g_{\tau\tau} + g_\tau^2) - \varepsilon_1 e - g g_\tau)^2}. \tag{17}$$

Therefore,  $\tau$  can be obtained by quadratures.

Note that to find  $k^2$ , we have to assume that

$$\tau(g g_{\tau\tau} + g_\tau^2) - \varepsilon_1 e - g g_\tau \neq 0.$$

Otherwise, we should also have that

$$\tau(d - \tau e - \varepsilon_1 g^2 + \varepsilon_1 \tau g g_\tau) - \varepsilon C(\varepsilon_1 g_\tau g + e) = 0.$$

Now, putting  $y(\tau) = g(\tau)^2$ , we have the system

$$\frac{\varepsilon_1}{2} (\tau^2 - \varepsilon C) y' - \varepsilon_1 \tau y + d\tau - e(\tau^2 + \varepsilon C) = 0, \quad \frac{\tau}{2} y'' - \varepsilon_1 e - \frac{y'}{2} = 0,$$

whose solutions are of the form  $y(\tau) = c_1 \tau^2 - 2\varepsilon_1 e \tau + \varepsilon_1 d - \varepsilon c_1 C$ , where  $c_1$  is an arbitrary constant. Therefore, the curvatures of the critical curves should satisfy

$$(\tau')^2 = \frac{\varepsilon_1 \varepsilon_2 k^2 (c_1 \tau^2 - 2\varepsilon_1 e \tau + \varepsilon_1 d - c_1 \varepsilon C)^2}{\varepsilon_1 d c_1 - c_1^2 \varepsilon C - e^2}.$$

It is worth pointing out that we obtain critical curves for any given curvature, getting a pretty wide family. However, this family does not appear in ref. [12], where the authors study the Lagrangian (1) in flat spaces.

#### 4. The solving natural equations problem

In order to find explicitly the critical curves of the Lagrangian  $\mathcal{L}(\gamma)$ , we look for a suitable coordinate system. To do that, we first give the following technical result

**Lemma 3.** *P and J commute.*

The method to obtain the coordinate system in flat spaces will be quite different from that followed when  $C \neq 0$ . However, as we will soon see, the parameters  $\lambda = d^2 - 4\varepsilon C e^2$  and  $|P \wedge J|$  seem to play an important role in the description of the particle path in any case. It is easy to show that they are related by

$$\lambda = (\langle P, P \rangle - \varepsilon C \langle J, J \rangle)^2 + 4C |P \wedge J|^2. \tag{18}$$

We begin with the description of the flat case.

##### 4.1. Flat case ( $C = 0$ )

We have to distinguish two cases according to the causal character of  $P$ . The method given in this case holds whenever  $P \neq 0$ , even when  $\langle J, J \rangle$  is constant.

##### 4.1.1. P is not null

Choose an orthonormal coordinate system  $(z_1, z_2, z_3)$  in  $\mathbb{R}_\nu^3$ , such that  $P$  and  $\partial_{z_1}$  are collinear and write  $P = u\partial_{z_1}$ . Let  $R_\theta = e^{\theta A}$  be the one-parameter group of rotations leaving invariant  $\partial_{z_1}$  (and  $P$ ), where

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varepsilon_r \\ 0 & -\varepsilon_\theta & 0 \end{pmatrix},$$

$\varepsilon_r = \langle \partial_{z_2}, \partial_{z_2} \rangle$  and  $\varepsilon_\theta = \langle \partial_{z_3}, \partial_{z_3} \rangle$ . Let

$$\psi(z, r, \theta) = R_\theta(z\partial_{z_1} + r\partial_{z_2})$$

be a cylindrical coordinate system.

Then  $\partial_z = \partial_{z_1}$ ,  $\partial_r = R_\theta(\partial_{z_2})$  and  $\partial_\theta = -\varepsilon_\theta r R_\theta(\partial_{z_3})$ . By using the properties of orthogonal matrices, we find that  $\partial_z$  and  $\partial_r$  are unit vectors (having the same causal character as  $P$  and  $\partial_{z_2}$ , respectively) and  $\langle \partial_\theta, \partial_\theta \rangle = \varepsilon_\theta r^2$ . On the other hand, we have

$$\gamma \wedge \partial_z = r\partial_r \wedge R_\theta(\partial_{z_1}) = -\varepsilon_\theta r R_\theta(\partial_{z_3}) = \partial_\theta.$$

Doing a translation, we get  $J = \alpha P + \gamma \wedge P$ , where  $\alpha = \frac{\langle J, P \rangle}{\langle P, P \rangle}$ . Therefore,

$$J = u(\alpha\partial_z + \partial_\theta)$$

so that

$$P \wedge J = \varepsilon_r \varepsilon_\theta r u^2 \partial_r \quad \text{and} \quad P \wedge T = u \left( -\frac{r_s}{r} \partial_\theta + \varepsilon_\theta \varepsilon_r r \theta_s \partial_r \right).$$

Hence, we get

$$r(s)^2 = \frac{\varepsilon_r |P \wedge J|^2}{\langle P, P \rangle^2}$$

or equivalently

$$r(s) = \frac{|P \wedge J|}{\sqrt{\lambda}}$$

and

$$\theta(s) = \int_0^s \frac{u \langle P \wedge T, P \wedge J \rangle}{|P \wedge J|^2} d\mu + c_1.$$

Finally, it is clear that

$$z(s) = \frac{u}{\langle P, P \rangle} \int_0^s \langle P, T \rangle d\mu + c_2.$$

The constants  $c_1$  and  $c_2$  determine, respectively, a rotation around  $P$  and a translation in the direction of  $P$ . This means that choosing two particular functions  $k$  and  $\tau$ , satisfying the Euler–Lagrange equations, all constants and freedom degrees are associated with rigid motions. This is an important point because we know that the coordinate expressions are necessary, but not sufficient. The fact of obtaining curves differing from rigid motions implies that all curves have the same curvatures and the critical curves are completely determined.

#### 4.1.2. $P$ is null

Proceeding as above, choose an orthonormal coordinate system  $(z_1, z_2, z_3)$  with  $\langle \partial_{z_1}, \partial_{z_1} \rangle = 1$  and  $\langle \partial_{z_2}, \partial_{z_2} \rangle = -1$  such that

$$P = \frac{1}{\sqrt{2}}(\partial_{z_1} - \partial_{z_2}).$$

We are going to see that this choice is always possible. Consider a Lorentzian plane  $\pi$  containing  $P$  and let  $\{w_1, w_2\}$  be an orthonormal frame of  $\pi$  such that  $w_1$  is spacelike and  $w_2$  timelike. Then, we can suppose that  $P = \mu(w_1 + w_2)$ , with  $\mu \in \mathbb{R}$ . The coordinate system verifying the above conditions is just  $\{e_1, e_2, e_3\}$ , where

$$e_1 = \frac{2\mu^2 + 1}{2\sqrt{2}\mu} w_1 + \frac{2\mu^2 - 1}{2\sqrt{2}\mu} w_2, \quad e_2 = \frac{1 - 2\mu^2}{2\sqrt{2}\mu} w_1 - \frac{2\mu^2 + 1}{2\sqrt{2}\mu} w_2,$$

and  $e_3$  is a unit vector orthogonal to the plane  $\pi$ . Now fix a one-parameter rotation group  $R_\theta = e^{\theta A}$  leaving invariant  $P$ , with

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -\varepsilon_\theta & -\varepsilon_\theta & 0 \end{pmatrix},$$

where  $\varepsilon_\theta = \langle \partial_{z_3}, \partial_{z_3} \rangle$ . Considering the pseudo-orthonormal basis

$$\{v_1 = \frac{1}{\sqrt{2}}(\partial_{z_1} - \partial_{z_2}), v_2 = \frac{1}{\sqrt{2}}(\partial_{z_1} + \partial_{z_2}), v_3 = \partial_{z_3}\},$$

we have  $A(v_1) = 0$ ,  $A(v_2) = -\varepsilon_\theta v_3$  and  $A(v_3) = v_1$ . On the other hand, taking the coordinate system  $\psi(z, r, \theta) = z v_1 + r R_\theta(v_2)$ , then the coordinate vector fields  $\partial_z = v_1$ ,  $\partial_r = R_\theta(v_2)$  and  $\partial_\theta = r R_\theta A(v_2) = -r \varepsilon_\theta R_\theta(v_3)$  form a pseudo-orthogonal basis and  $\{\partial_z, \partial_r, \frac{1}{r} \partial_\theta\}$  is a pseudo-orthonormal one having  $\partial_z$  and  $\partial_r$  as null vectors such that  $\langle \partial_z, \partial_r \rangle = 1$ . Moreover,

$$\gamma \wedge v_1 = r R_\theta(v_2) \wedge R_\theta(v_1) = r R_\theta(v_2 \wedge v_1) = -\varepsilon_\theta r R_\theta(v_3) = \partial_\theta.$$

Doing a translation, we get  $J = \epsilon v_2 + \gamma \wedge P$ . Now, we are going to find  $v_2$  in terms of the coordinate vector fields. To do that, observe that

$$R_\theta(v_3) = \theta v_1 + v_3 \quad \text{and} \quad R_\theta(v_2) = -\epsilon_\theta \frac{\theta^2}{2} v_1 + v_2 - \epsilon_\theta \theta v_3.$$

Then

$$v_2 = -\epsilon_\theta \frac{\theta^2}{2} P + \partial_r - \frac{\theta}{r} \partial_\theta \quad \text{and} \quad J = -\epsilon_\theta e \frac{\theta^2}{2} P + e \partial_r + \left(1 - e \frac{\theta}{r}\right) \partial_\theta.$$

Therefore, developing  $\langle J, T \rangle$ ,  $\langle J, J \rangle$  and  $\langle P, T \rangle$  we find out the coordinates  $z$ ,  $\theta$  and  $r$  as

$$z(s) = \epsilon_\theta \left( \frac{\theta^2 r}{2} - \frac{\theta r^2}{e} + \frac{r^3}{3e^2} \right) - \frac{1}{e^2} \int \langle J \wedge P, T \wedge J \rangle d\mu + c_1,$$

$$\theta(s) = \frac{1}{2er} (r^2 - \epsilon_\theta \langle J, J \rangle), \quad r(s) = \int \langle P, T \rangle d\mu + c_2.$$

When  $e = 0$ , as  $\langle T, T \rangle = 2z_s r_s + \epsilon_\theta \theta_s^2 r^2$ , we get

$$z(s) = - \int \frac{|J \wedge T|^2}{2\langle P, T \rangle \langle J, J \rangle} d\mu + c_1, \quad \theta(s) = - \int \frac{\langle J, T \rangle}{\langle J, J \rangle} d\mu + c_2, \quad r(s)^2 = \langle J, J \rangle.$$

Obviously, the integration constants  $c_1$  and  $c_2$  produce rigid motions, so that the critical curves are completely determined. Indeed,  $c_1$  gives the translation  $c_1 v_1$ , whereas  $c_2$  produces the translation  $\frac{-\epsilon_\theta c_2^3}{6e^2} v_1 + c_2 v_2 - \frac{\epsilon_\theta c_2^2}{2e} v_3$  followed by the rotation  $R_{c_2/e}$ . This rigid motion is associated with the expression of  $J$  in terms of  $P$ , which is not invariant by rotations, so that we must apply a translation to compensate this fact.

#### 4.2. Non-flat case ( $C \neq 0$ )

The manifold  $M_\nu^3(C)$  can be viewed as a hyperquadric in  $\mathbb{R}_\mu^4$ , where  $\mu = \nu$  or  $\mu = \nu + 1$ , according to  $C > 0$  or  $C < 0$ , respectively. Choose the following parametrization

$$X(\theta, \varphi, \psi) = e^{\theta A} e^{\varphi B} c(\psi),$$

where  $A, B \in \mathfrak{o}(4, \mu)$  commute,  $c(\psi) = (a_1(\psi), a_2(\psi), a_3(\psi), a_4(\psi))^t$  is a curve in  $\mathbb{R}_\mu^4$  satisfying  $\langle c(\psi), c(\psi) \rangle = 1/C$  and  $t$  denotes transpose. It is easy to see that the coordinate vector fields are given by

$$X_\theta = A e^{\theta A} e^{\varphi B} c(\psi),$$

$$X_\varphi = B e^{\theta A} e^{\varphi B} c(\psi),$$

$$X_\psi = e^{\theta A} e^{\varphi B} c'(\psi),$$

and satisfy

$$g_{\theta\theta} \equiv \langle X_\theta, X_\theta \rangle = -c(\psi)^t \Lambda A^2 c(\psi), \quad g_{\varphi\varphi} \equiv \langle X_\varphi, X_\varphi \rangle = -c(\psi)^t \Lambda B^2 c(\psi),$$

$$g_{\psi\psi} \equiv \langle X_\psi, X_\psi \rangle = c'(\psi)^t \Lambda c'(\psi), \quad g_{\theta\varphi} \equiv \langle X_\theta, X_\varphi \rangle = -c(\psi)^t \Lambda A B c(\psi),$$

$$g_{\theta\psi} \equiv \langle X_\theta, X_\psi \rangle = -c(\psi)^t \Lambda A c'(\psi), \quad g_{\varphi\psi} \equiv \langle X_\varphi, X_\psi \rangle = -c(\psi)^t \Lambda B c'(\psi),$$

where  $\Lambda$  stands for the diagonal matrix  $\text{diag}[\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4]$  representing the canonical metric in  $\mathbb{R}_\mu^4$ .

Now, we have to distinguish according to the isometry Lie algebra is either  $\mathfrak{o}(4, \delta)$ , with  $\delta = 0, 1$  or  $\mathfrak{o}(4, 2)$ .

4.2.1. The isometry Lie algebra is  $\mathfrak{o}(4, \delta)$ ,  $\delta = 0, 1$

We can assume that  $\epsilon_2 = \epsilon_3 = \epsilon_4 = 1$  and  $\epsilon_1 = \epsilon$ , where  $\epsilon$  stands for  $\det(\Lambda)$ . In order to find explicitly the coordinate functions of the critical curves we have to consider two cases: (1)  $d^2 + e^2 \neq 0$  or (2)  $d = e = 0$ .

**Case 1: ( $d^2 + e^2 \neq 0$ ).** Then choose  $A = P_1$  and  $B = L_1$  (as in Appendix A.1) and  $c(\psi) = (a_1(\psi), 0, a_3(\psi), 0)^t$ . It is easy to show that the coordinate vector fields form an orthogonal system with  $g_{\theta\theta} = a_1(\psi)^2$  and  $g_{\varphi\varphi} = a_3(\psi)^2$ . Now, as  $P$  and  $J$  commute, we can apply a rotation, when necessary (see Appendices A.1 and A.2), to write  $P$  and  $J$  as

$$\left. \begin{aligned} P &= p_1 X_\theta + p_2 X_\varphi \\ J &= q_1 X_\theta + q_2 X_\varphi \end{aligned} \right\}, \quad \text{that is,} \quad \begin{pmatrix} P \\ J \end{pmatrix} = M \begin{pmatrix} X_\theta \\ X_\varphi \end{pmatrix}, \quad M = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}.$$

From (8), we deduce that

$$\left. \begin{aligned} p_1^2 + \epsilon C q_1^2 &= \epsilon C d, & p_2^2 + \epsilon C q_2^2 &= C d, \\ p_1 q_1 &= \epsilon C e, & p_2 q_2 &= C e. \end{aligned} \right\} \tag{19}$$

Writing down  $\langle J, J \rangle$  in terms of  $q_i$ , we find that

$$a_1(\psi(s))^2 = \frac{C\langle J, J \rangle - q_2^2}{C(q_1^2 - \epsilon q_2^2)} \quad \text{and} \quad a_3(\psi(s))^2 = \frac{q_1^2 - \epsilon C\langle J, J \rangle}{C(q_1^2 - \epsilon q_2^2)}. \tag{20}$$

Observe that  $q_1^2 - \epsilon q_2^2 \neq 0$ , because  $\langle J, J \rangle$  is not constant. On the other hand, we can write  $X_\theta$  and  $X_\varphi$  in terms of  $P$  and  $J$  as

$$\left. \begin{aligned} X_\theta &= \alpha_1 P + \alpha_2 J \\ X_\varphi &= \beta_1 P + \beta_2 J \end{aligned} \right\}, \quad \text{where} \quad \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} = M^{-1}.$$

Finally, as  $T = \theta'(s)X_\theta + \varphi'(s)X_\varphi + \psi'(s)X_\psi$ , from (20) we conclude

$$\theta(s) = \int \frac{\langle T, X_\theta \rangle}{g_{\theta\theta}} d\mu = C(q_1^2 - \epsilon q_2^2) \int \frac{\alpha_1 \langle P, T \rangle + \alpha_2 \langle J, T \rangle}{C\langle J, J \rangle - q_2^2} d\mu + c_1$$

and

$$\varphi(s) = \int \frac{\langle T, X_\varphi \rangle}{g_{\varphi\varphi}} d\mu = C(q_1^2 - \epsilon q_2^2) \int \frac{\beta_1 \langle P, T \rangle + \beta_2 \langle J, T \rangle}{q_1^2 - \epsilon C\langle J, J \rangle} d\mu + c_2.$$

There are several choices for the signs of  $q_1$  and  $q_2$ , but only one can be chosen, since the corresponding critical curves differ each others from rotations. Now we have to distinguish two subcases:  $\epsilon = 1$  and  $\epsilon = -1$ .

**Case 1.1 ( $\epsilon = 1$ ).** Then the hyperquadric is  $\mathbb{S}^3(C)$ , so that  $\epsilon = 1$ . It is easy to see that  $d > 0$  and  $\lambda > 0$ , unless  $\langle J, J \rangle$  is constant. As before, we can suppose that  $q_1 > q_2 \geq 0$ . A straightforward computation shows that

- $p_1 = \sqrt{C}q_2$  and  $p_2 = \sqrt{C}q_1$  when  $e > 0$ ; and
- $p_1 = -\sqrt{C}q_2$  and  $p_2 = -\sqrt{C}q_1$  when  $e < 0$ .

**Case 1.2 ( $\epsilon = -1$ ).** Then the hyperquadric is either  $\mathbb{S}_1^3(C)$  or  $\mathbb{H}^3(C)$ , according to  $C > 0$  or  $C < 0$ , respectively. A similar computation as above shows that

- $p_1 = \sqrt{-\varepsilon C}q_2$  and  $p_2 = -\sqrt{-\varepsilon C}q_1$  when  $\varepsilon e > 0$ ; and
- $p_1 = -\sqrt{-\varepsilon C}q_2$  and  $p_2 = \sqrt{-\varepsilon C}q_1$  when  $\varepsilon e < 0$ .

**Case 2: ( $d = e = 0$ ).** Take  $A = L_2 + P_3$ ,  $B = L_3 - P_2$  and  $c(\psi) = (a_1(\psi), a_2(\psi), 0, 0)^t$ . Then the coordinate vector fields form an orthogonal frame with  $g_{\theta\theta} = g_{\varphi\varphi} = (a_1(\psi) + a_2(\psi))^2$ . In this case, we can suppose that  $P = X_\theta$  and  $J = q_1 X_\theta + q_2 X_\varphi$ . From (8), we deduce that  $q_1 = 0$  and  $q_2^2 = -1/(\varepsilon C)$ , so that  $(a_1(\psi) + a_2(\psi))^2 = -\varepsilon C \langle J, J \rangle$ . Without loss of generality, we can assume that  $q_2 > 0$ . Finally, as  $T = \theta'(s)X_\theta + \varphi'(s)X_\varphi + \psi'(s)X_\psi$ , we deduce that

$$\theta(s) = -\frac{1}{\varepsilon C} \int \frac{\langle P, T \rangle}{\langle J, J \rangle} d\mu + c_1,$$

$$\varphi(s) = q_2 \int \frac{\langle J, T \rangle}{\langle J, J \rangle} d\mu + c_2.$$

*Explicit expression of the coordinate system X*

We have to distinguish several cases:

- (a)  $\varepsilon = 1$ , then  $M_v^3(C) = \mathbb{S}^3(C)$ . Take  $a_1(\psi) = \frac{1}{\sqrt{C}} \sin \psi$  and  $a_3(\psi) = \frac{1}{\sqrt{C}} \cos \psi$ , and then

$$X(\theta, \varphi, \psi) = \frac{1}{\sqrt{C}} (\cos \theta \sin \psi, -\sin \theta \sin \psi, \cos \varphi \cos \psi, -\sin \varphi \cos \psi).$$

- (b)  $\varepsilon = -1$ , then  $M_v^3(C) = \mathbb{S}_1^3(C)$  or  $\mathbb{H}^3(C)$ . Now, we have two subclasses:

- (b1)  $d^2 + e^2 \neq 0$ . Then

$$X(\theta, \varphi, \psi) = (a_1(\psi) \cosh \theta, a_1(\psi) \sinh \theta, a_3(\psi) \cosh \varphi, -a_3(\psi) \sinh \varphi).$$

- (b2)  $d = e = 0$ . Then

$$X(\theta, \varphi, \psi) = (a_1(\psi) + \frac{1}{2}(\varphi^2 + \theta^2)(a_1(\psi) + a_2(\psi)), \\ a_2(\psi) - \frac{1}{2}(\varphi^2 + \theta^2)(a_1(\psi) + a_2(\psi)), -\varphi(a_1(\psi) + a_2(\psi)), \theta(a_1(\psi) + a_2(\psi))).$$

The functions  $a_1(\psi)$ ,  $a_2(\psi)$  and  $a_3(\psi)$  are given by

- $a_1(\psi) = \frac{1}{\sqrt{C}} \sinh \psi$  and  $a_2(\psi) = a_3(\psi) = \frac{1}{\sqrt{C}} \cosh \psi$  in  $\mathbb{S}_1^3(C)$ .
- $a_1(\psi) = \frac{1}{\sqrt{-C}} \cosh \psi$  and  $a_2(\psi) = a_3(\psi) = \frac{1}{\sqrt{-C}} \sinh \psi$  in  $\mathbb{H}^3(C)$ .

Finally, observe that there are interesting relationships among  $a_1$ ,  $a_2$  (or  $a_3$ ) and  $|P \wedge J|$ . In cases (a) and (b1), we have

$$|P \wedge J| = \sqrt{|C\lambda|} a_1(\psi) a_3(\psi) = \begin{cases} \frac{\sqrt{|C\lambda|}}{2C} \sin 2\psi & \text{in (a)} \\ \frac{\sqrt{|C\lambda|}}{2|C|} \sinh 2\psi & \text{in (b1)} \end{cases}$$

In case (b2), the relation is

$$|P \wedge J| = \frac{1}{\sqrt{|C|}} (a_1(\psi) + a_2(\psi))^2 = \frac{1}{|C|^{3/2}} e^{2\psi}.$$

4.2.2. The isometry Lie algebra is  $\mathfrak{o}(4, 2)$

First, we can suppose that  $\varepsilon = -1$  and  $C < 0$ , since the anti-de Sitter space is the only space form whose Lie algebra is  $\mathfrak{o}(4, 2)$ . We will take  $a_2(\psi) = \frac{1}{\sqrt{\varepsilon C}} \cosh \psi$  and  $a_3(\psi) = \frac{1}{\sqrt{\varepsilon C}} \sinh \psi$ , and choose the matrices  $A$  and  $B$  as in Appendix A.2, having nine possible cases. In any of them, the products  $g_{\theta\theta}$  and  $g_{\varphi\varphi}$  are constant (more precisely, they take values in  $\{0, -\frac{1}{C}, \frac{1}{C}\}$ ). However,  $g_{\theta\varphi}$  is a linear combination of  $\cosh(2\psi)$  and  $\sinh(2\psi)$ , and therefore it depends on  $\psi$  in such a way that it is not constant anywhere. Furthermore,  $g_{\theta\psi}$  and  $g_{\varphi\psi}$  both vanish. From Appendix A.2, along with  $[P, J] = 0$ , we deduce that

$$\begin{pmatrix} P \\ J \end{pmatrix} = M \begin{pmatrix} X_\theta \\ X_\varphi \end{pmatrix}.$$

The coefficients  $q_1$  and  $q_2$  do not vanish, because  $\langle J, J \rangle$  is not constant. From (8), we deduce that

$$\begin{aligned} p_1 p_2 + \varepsilon C q_1 q_2 &= 0, & (p_1^2 + \varepsilon C q_1^2) g_{\theta\theta} + (p_2^2 + \varepsilon C q_2^2) g_{\varphi\varphi} &= d, \\ p_1 q_2 + p_2 q_1 &= 0, & q_1 p_1 g_{\theta\theta} + q_2 p_2 g_{\varphi\varphi} &= e. \end{aligned} \tag{21}$$

Now  $\lambda = d^2 - 4\varepsilon C e^2 = (d + 2\sqrt{\varepsilon C} e)(d - 2\sqrt{\varepsilon C} e)$  can be positive, negative or zero. We shall take, without loss of generality,  $p_1$  with the same sign of  $q_1$  and then  $p_2$  and  $q_2$  will have opposite sign. Moreover, when  $\lambda < 0$  we can suppose  $q_1 > 0$  and take  $q_2$  with the same sign of  $-(d + 2C\langle J, J \rangle)$ . If  $\lambda > 0$ , we can assume that  $q_1$  and  $q_2$  are positive.

From the left-hand equations in (21), we deduce that  $p_1 = \sqrt{\varepsilon C} q_1$  and  $p_2 = -\sqrt{\varepsilon C} q_2$ . Furthermore, considering the right-hand equations, we get

$$4\varepsilon C g_{\theta\theta} q_1^2 = d + 2\sqrt{\varepsilon C} e \quad \text{and} \quad 4\varepsilon C g_{\varphi\varphi} q_2^2 = d - 2\sqrt{\varepsilon C} e. \tag{22}$$

The parameters  $d + 2\sqrt{\varepsilon C} e$  and  $d - 2\sqrt{\varepsilon C} e$  determine the choices of  $A$  and  $B$ . Thus, we shall study separately the cases  $\lambda \neq 0$  and  $\lambda = 0$ .

**Case 1: ( $\lambda \neq 0$ ).** Now  $d + 2\sqrt{\varepsilon C} e$  and  $d - 2\sqrt{\varepsilon C} e$  do not vanish. A direct computation yields  $g_{\theta\theta} = \varepsilon_\theta / \varepsilon C$  and  $g_{\varphi\varphi} = \varepsilon_\varphi / \varepsilon C$ .

On the other hand, from (8), we get

$$g_{\theta\varphi} = \frac{2\varepsilon C \langle J, J \rangle - d}{4\varepsilon C q_1 q_2}. \tag{23}$$

From the relationships among  $P, J, X_\theta$  and  $X_\varphi$ , it is easy to see that

$$\begin{pmatrix} X_\theta \\ X_\varphi \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} P \\ J \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} = M^{-1}, \tag{24}$$

and therefore

$$X_\theta \wedge X_\varphi = \frac{1}{\det(M)} P \wedge J = \frac{1}{2p_1 q_2} P \wedge J.$$

From these equations and  $T = \theta'(s)X_\theta + \varphi'(s)X_\varphi + \psi'(s)X_\psi$  it can be deduced that

$$\begin{aligned} \theta'(s) &= \frac{\langle T \wedge X_\varphi, X_\theta \wedge X_\varphi \rangle}{|X_\theta \wedge X_\varphi|^2}, \\ \varphi'(s) &= -\frac{\langle T \wedge X_\theta, X_\theta \wedge X_\varphi \rangle}{|X_\theta \wedge X_\varphi|^2}. \end{aligned}$$

Then, from (24), we get

$$\theta(s) = 2q_1q_2 \int \frac{\langle T \wedge (\beta_1 P + \beta_2 J), P \wedge J \rangle}{|P \wedge J|^2} d\mu$$

and

$$\varphi(s) = -2q_1q_2 \int \frac{\langle T \wedge (\alpha_1 P + \alpha_2 J), P \wedge J \rangle}{|P \wedge J|^2} d\mu.$$

A straightforward computation yields

$$g_{\theta\varphi} = \begin{cases} 2a_2(\psi)a_3(\psi), & \text{if } \lambda < 0, \\ a_2(\psi)^2 + a_3(\psi)^2, & \text{if } \lambda > 0. \end{cases}$$

These expressions, together with  $a_2^2 - a_3^2 = \frac{1}{\varepsilon C}$ , (18), (22) and (23), lead us to the following:

(a) when  $\lambda < 0$

$$a_2(\psi(s))^2 = \frac{|P \wedge J|^2 - \sqrt{-\lambda}}{2\varepsilon C\sqrt{-\lambda}} \quad \text{and} \quad a_3(\psi(s))^2 = \frac{|P \wedge J|^2 + \sqrt{-\lambda}}{2\varepsilon C\sqrt{-\lambda}},$$

from which we deduce that

$$|P \wedge J| = \frac{\sqrt{-\varepsilon C\lambda}}{2}(a_2^2 + a_3^2) = \frac{\sqrt{-\varepsilon C\lambda}}{2\varepsilon C} \cosh 2\psi;$$

(b) when  $\lambda > 0$

$$a_2(\psi(s))^2 = \frac{|2\varepsilon C\langle J, J \rangle - d| - \sqrt{\lambda}}{2\varepsilon C\sqrt{\lambda}} \quad \text{and} \quad a_3(\psi(s))^2 = \frac{|2\varepsilon C\langle J, J \rangle - d| + \sqrt{\lambda}}{2\varepsilon C\sqrt{\lambda}},$$

so that we get

$$|P \wedge J| = \sqrt{\varepsilon C\lambda}a_2a_3 = \frac{\sqrt{\varepsilon C\lambda}}{2\varepsilon C} \sinh 2\psi.$$

**Case 2: ( $\lambda = 0$ ).** We have to distinguish two subcases depending on the values of  $\{d + 2\sqrt{\varepsilon C}e$  and  $d - 2\sqrt{\varepsilon C}e\}$ .

**Subcase 2.1.**  $d + 2\sqrt{\varepsilon C}e = 0$  and  $d - 2\sqrt{\varepsilon C}e \neq 0$ . Then  $q_1 = 1$  (see Appendix A.2) and  $g_{\theta\theta} = 0$ ,  $g_{\varphi\varphi} = \varepsilon_\varphi/(\varepsilon C)$ ,  $g_{\theta\varphi} = (a_2 + a_3)^2$  and  $q_2^2 = \frac{1}{2}\varepsilon_\varphi d$ . Now, from

$$g_{\theta\varphi} = \frac{2\varepsilon C\langle J, J \rangle - d}{4\varepsilon Cq_2}$$

we deduce that

$$a_2(\psi) = \frac{g_{\theta\varphi} + \frac{1}{\varepsilon C}}{2\sqrt{g_{\theta\varphi}}} \quad \text{and} \quad a_3(\psi) = \frac{g_{\theta\varphi} - \frac{1}{\varepsilon C}}{2\sqrt{g_{\theta\varphi}}}.$$

Therefore,

$$|P \wedge J| = \sqrt{2\varepsilon C\varepsilon_\varphi d}(a_2 + a_3)^2 = \frac{\sqrt{2\varepsilon C\varepsilon_\varphi d}}{\varepsilon C} e^{2\psi}.$$



The expressions for  $\theta$  and  $\varphi$  agree with that obtained when  $\lambda \neq 0$ , taking  $q_1 = 1$ , but  $\varphi$  can now be simplified as

$$\varphi(s) = 2\varepsilon\sqrt{\varepsilon C}q_2 \int \frac{(\langle P, T \rangle + \sqrt{\varepsilon C}\langle J, T \rangle)}{2\varepsilon C\langle J, J \rangle + d} ds.$$

Note that the symmetric case  $d + 2\sqrt{\varepsilon C}e \neq 0$  and  $d - 2\sqrt{\varepsilon C}e = 0$  only differs from this in a rotation.

**Subcase 2.2.**  $d + 2\sqrt{\varepsilon C}e = 0$  and  $d - 2\sqrt{\varepsilon C}e = 0$ . Then,  $d = e = 0$  and  $g_{\theta\theta} = g_{\varphi\varphi} = 0$ ,  $q_1 = q_2 = 1$  and  $g_{\theta\varphi} = 2(a_2 + a_3)^2$ . On the other hand,  $g_{\theta\varphi} = \frac{1}{2}\langle J, J \rangle$ , so we have

$$a_2(\psi(s)) = \frac{\varepsilon C\langle J, J \rangle + 4}{\varepsilon C\sqrt{2}\langle J, J \rangle} \quad \text{and} \quad a_3(\psi(s)) = \frac{\varepsilon C\langle J, J \rangle - 4}{\varepsilon C\sqrt{2}\langle J, J \rangle}.$$

In this case,

$$|P \wedge J| = 4\sqrt{\varepsilon C}(a_2 + a_3)^2 = \frac{4\sqrt{\varepsilon C}}{\varepsilon C}e^{2\psi}.$$

Again the expressions of  $\theta$  and  $\varphi$ , when  $\lambda \neq 0$ , hold, but they can be simplified as

$$\theta(s) = \int \frac{-\langle P, T \rangle + \sqrt{\varepsilon C}\langle J, T \rangle}{\sqrt{\varepsilon C}\langle J, J \rangle} \quad \text{and} \quad \varphi(s) = \int \frac{\langle P, T \rangle + \sqrt{\varepsilon C}\langle J, T \rangle}{\sqrt{\varepsilon C}\langle J, J \rangle}.$$

*Explicit expression of the coordinate system X*

Finally, we are going to describe how to construct the coordinate systems. In [Appendix A.2](#), we have seen that the possible choices of the matrices  $A$  and  $B$  are  $\xi_1^\delta = \sqrt{2}P_1^\delta$ ,  $\xi_{-1}^\delta = \sqrt{2}P_2^\delta$  and  $\xi_0^\delta = \sqrt{2}(P_1^\delta + P_2^\delta)$ . Here,  $\delta = 1$  corresponds to the choices of  $A$  and  $\delta = -1$  with those of  $B$ . Moreover, the lower index in  $\xi$  gives the causal character of  $X_\theta$  for the choices of  $A$  and the casual character of  $X_\varphi$  for those of  $B$ . Now, we observe that

$$e^{\omega\xi_1^\delta} = \begin{pmatrix} \cos \omega & \sin \omega & 0 & 0 \\ \sin \omega & \cos \omega & 0 & 0 \\ 0 & 0 & \cos \omega & \delta \sin \omega \\ 0 & 0 & \delta \sin \omega & \cos \omega \end{pmatrix},$$

$$e^{\omega\xi_{-1}^\delta} = \begin{pmatrix} \cosh \omega & 0 & \sinh \omega & 0 \\ 0 & \cosh \omega & 0 & -\delta \sinh \omega \\ \sinh \omega & 0 & \cosh \omega & 0 \\ 0 & -\delta \sinh \omega & 0 & \cosh \omega \end{pmatrix},$$

$$e^{\omega\xi_0^\delta} = \begin{pmatrix} 1 & \omega & \omega & 0 \\ -\omega & 1 & 0 & -\delta\omega \\ \omega & 0 & 1 & \delta\omega \\ 0 & -\delta\omega & -\delta\omega & 1 \end{pmatrix}.$$

Then by computing  $(e^{\theta A} e^{\varphi B} c(\psi))$ , we obtain the coordinate systems. For example,

(a) when  $X_\theta$  and  $X_\varphi$  are spacelike, the coordinate system  $X(\theta, \varphi, \psi)$  is given by

$$\frac{1}{\sqrt{\varepsilon C}} (\cosh \psi \sin(\theta + \varphi), \cosh \psi \cos(\theta + \varphi), \sinh \psi \cos(\theta - \varphi), -\sinh \psi \sin(\theta - \varphi))$$

(b) when they are timelike, the coordinate system is

$$\frac{1}{\sqrt{\varepsilon C}} (\sinh \psi \sinh(\theta + \varphi), \cosh \psi \cosh(\theta - \varphi), \sinh \psi \cosh(\theta + \varphi), -\cosh \psi \sinh(\theta - \varphi)).$$

## 5. Conclusions

In ref. [11], we have studied actions in  $D = 3$  spacetimes whose Lagrangian is a linear function  $m + nk + p\tau$  on the curvature and torsion of the particle path, finding out that trajectories are Lancret curves, or generalized helices. Indeed, the critical curves are always Lancret curves, which are obtained by geometrical integration involving the Hopf fibrations (see also ref. [5]). Here, we go further in a two-fold sense, assuming that the Lagrangian density is an arbitrary function on the curvature and torsion of the particle path which is lying in a three-dimensional pseudo-Riemannian space form. We have got two Killing vector fields along curves  $P$  and  $J$  and exploited the machinery supplied by them, which became a fruitful tool in our earlier and recent paper. Actually, the integral equations are reduced to a system involving  $P$  and  $J$ , which is equivalent to the Euler–Lagrange equations if, and only if,  $\langle J, J \rangle$  is not constant. Then we have solved the motion equations and found out solutions which, as a pretty interesting fact, are not generalized helices. We note that when the Lagrangian density is  $m + nk + p\tau$ , then  $\langle J, J \rangle$  is constant.

To obtain explicitly the critical curves of the Lagrangian, we have chosen suitable coordinate frames where the Frenet equations have been integrated. With the help of the corresponding Lie algebras, a complete system of solutions is given in the de Sitter  $\mathbb{S}_1^3$  and anti de Sitter  $\mathbb{H}_1^3$  worlds as well as in the non-flat Riemannian space forms  $\mathbb{S}^3$  and  $\mathbb{H}^3$ .

Finally, an open and interesting problem could be the searching for critical curves different from the generalized helices when  $\langle J, J \rangle$  is constant, as well as a suitable method to get them.

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## Appendix A

We look for characteristic elements of the orbits of the Lie algebra  $\mathfrak{o}(4, \nu)$ , when  $\nu = 0, 1, 2$ . That is, given  $A \in \mathfrak{o}(4, \nu)$ , we apply a rotation  $G$  to choose a new coordinate system where the matrix associated to  $A$  is  $\bar{A} = GAG^{-1}$ . Thus is obtained the orbit of  $A$ , where we wish to find an element in a simple way. Now, there is a bijective mapping  $\Phi : \mathfrak{o}(4, \nu) \rightarrow \mathcal{K}$  such that

$\Phi([A, B]) = -[\Phi(A), \Phi(B)]$ , where  $\mathcal{K}$  is the subset of Killing vector fields (see ref. [14]). Then, by means of  $\Phi$ , we use characteristic elements of the Lie algebra to find characteristic ones of  $\mathcal{K}$  (see ref. [15] for details).

A.1.  $\mathfrak{o}(4, 1)$

We will show that any element of the Lie algebra  $\mathfrak{o}(4, 1)$  is given by a linear combination of the vector fields  $X_\theta$  and  $X_\varphi$ .

Let  $A_{ij}$  be a matrix such that  $a_{ij} = 1$  and 0 the remaining entries. Let us define

$$M_{ij} = A_{ij} - \varepsilon_i \varepsilon_j A_{ji},$$

where  $\varepsilon_0 = -1$  and  $\varepsilon_i = 1, i = 1, 2, 3$ . A basis of  $\mathfrak{o}(4, 1)$  is given by

$$P_i = M_{0i}, \quad L_1 = M_{23}, \quad L_2 = M_{31}, \quad L_3 = M_{12}.$$

It is not difficult to see that

$$[P_i, P_j] = \sum_k \varepsilon_{ij}^k L_k, \quad [L_i, L_j] = -\sum_k \varepsilon_{ij}^k L_k, \quad [L_i, P_j] = -\sum_k \varepsilon_{ij}^k P_k,$$

where  $\varepsilon_{ij}^k = 1$ , when  $\{i, j, k\}$  is an even permutation of  $\{1, 2, 3\}$ , and  $-1$  otherwise.

Fix  $A \in \mathfrak{o}(4, 1)$  and define the orbit

$$\text{orb}(A) = \{B \in \mathfrak{o}(4, 1) : B = GAG^{-1}, \text{ for any } G = \prod_{i \in I} \exp(X_i) \text{ and } X_i \in \mathfrak{o}(4, 1), \forall i \in I\},$$

$I$  being a finite subset. It is clear that  $\mathfrak{o}(4, 1)$  is a disjoint union of orbits. To determine the orbit of any  $A$ , first observe that when

$$A_L(s) = \exp(sL)A \exp(-sL),$$

where  $L \in \mathfrak{o}(4, 1)$ , then

$$A'_L(s) = [L, A_L(s)].$$

Therefore, writing  $A_L(s)$  and  $L$  in the above basis of  $\mathfrak{o}(4, 1)$

$$A_L(s) = \sum_i l_i(s)L_i + \sum_i p_i(s)P_i, \quad L = \sum_i \alpha_i L_i + \sum_i \beta_i P_i,$$

the following equations hold

$$\begin{cases} l'_k = -\sum_{i,j} \varepsilon_{i,j}^k (\alpha_j l_i - \beta_j p_i), \\ p'_k = -\sum_{i,j} \varepsilon_{i,j}^k (\alpha_j p_i + \beta_j l_i). \end{cases}$$

It is not difficult to see that the Casimir functions for  $\mathfrak{o}(4, 1)$ , with the initial condition  $A_L(0) = A$ , satisfy

$$\sum_k l_k p_k = k_1, \quad \sum_k l_k^2 - \sum_k p_k^2 = k_2,$$

for certain constants  $k_1$  and  $k_2$ . Then  $\text{orb}(A)$  is lying in a certain level set  $\mathcal{C} \subset \mathfrak{o}(4, 1)$  of the Casimir functions. To see that they exactly agree we have to show that  $\mathcal{C}$  is connected and  $\text{orb}(A)$  is open. As  $\mathcal{C}$  is a disjoint union of its orbits, the connectedness will mean that it contains a single orbit.

To prove that  $\mathcal{C}$  is connected, we consider three-dimensional slices  $p = (p_1, p_2, p_3) = \text{const}$ . They are circles obtained by the intersection of a normal plane at  $p$  with a sphere of radius  $\sqrt{k_2 + \langle p, p \rangle}$ , i.e.,

$$\langle l, p \rangle = k_1, \quad \langle l, l \rangle = k_2 + \langle p, p \rangle,$$

where  $l = (l_1, l_2, l_3)$ . To compute the values of  $p$  where the slice is not empty we have to write the condition that the distance from the plane to the origin is lower than the radius of the sphere, that is,

$$k_2 + \langle p, p \rangle \geq \frac{k_1^2}{\langle p, p \rangle}.$$

This is a connected set and so is  $\mathcal{C}$ . It is clear that, at each orbit except for the case  $k_1 = k_2 = 0$ , there is at least a representative as  $L = aL_1 + bP_1$ . If  $k_1 = k_2 = 0$ , we can choose a representative element  $L_2 + P_3$ . Finally, by choosing a non zero element of the Lie algebra of the form  $L = aL_1 + bP_1$ , any other element commuting with  $L$  must be a linear combination of  $L_1$  and  $P_1$ . Analogously we find that the elements commuting with  $L_2 + P_3$  are of the form  $a(L_2 + P_3) + b(L_3 - P_2)$

### A.2. $\mathfrak{o}(4, 0)$ and $\mathfrak{o}(4, 2)$

We will see now that  $\mathfrak{o}(4, 2\varrho)$ ,  $\varrho = 0, 1$ , is isomorphic to the product  $\mathfrak{o}(3, \varrho) \times \mathfrak{o}(3, \varrho)$ . With the above notation, we set

$$L_1^{\delta_1} = (M_{01} + \delta_1 M_{23}), \quad L_2^{\delta_2} = (M_{02} + \delta_2 M_{13}), \quad L_3^{\delta_3} = (M_{12} + \delta_3 M_{30}),$$

where  $\delta_i = \pm 1$ ,  $i = 1, 2, 3$ . When  $\varepsilon_0 = \varepsilon_1 = 1$  and  $\varepsilon_2 = \varepsilon_3 = \varepsilon$ , we obtain

$$[L_1^{\delta_1}, L_2^{\delta_2}] = (1 - \delta_1 \delta_2) L_3^{\varepsilon \delta_2}, \quad [L_2^{\delta_2}, L_3^{\delta_3}] = -\varepsilon(1 + \varepsilon \delta_2 \delta_3) L_1^{-\delta_2},$$

$$[L_3^{\delta_3}, L_1^{\delta_1}] = -(1 - \varepsilon \delta_1 \delta_3) L_2^{-\delta_1}.$$

Therefore, if  $\delta^2 = 1$ , set

$$P_1^\delta = \frac{1}{\sqrt{2}} L_1^\delta, \quad P_2^\delta = \frac{1}{\sqrt{2}} L_2^{-\delta}, \quad P_3^\delta = \frac{1}{\sqrt{2}} L_3^{-\varepsilon \delta},$$

to get

$$[P_1^\delta, P_2^\delta] = -\varepsilon P_3^\delta, \quad [P_2^\delta, P_3^\delta] = -\varepsilon P_1^\delta, \quad [P_3^\delta, P_1^\delta] = -P_2^\delta.$$

Moreover,  $P_i^\delta$  and  $P_j^{-\delta}$  always commute. Then  $\mathfrak{o}(4, 2\varrho)$  splits as a direct sum of two commuting subalgebras

$$\mathfrak{o}(4, 2\varrho) = E_1 \oplus E_{-1},$$

where

$$E_1 = \text{span}\{P_1^1, P_2^1, P_3^1\} \quad \text{and} \quad E_{-1} = \text{span}\{P_1^{-1}, P_2^{-1}, P_3^{-1}\}.$$

Both subalgebras can be identified with  $\mathfrak{o}(3, \varrho)$ . We look for canonical elements obtained from rigid motions. We will study separately each direct summand, because when we apply a rotation generated by an element of  $E_1$ , the component in  $E_{-1}$  is left invariant and viceversa. To determine  $E_\mu$ ,  $\mu = -1, 1$ , take  $A \in E_\mu$ . If

$$A_L(s) = \exp(sL)A \exp(-sL),$$

then

$$A'_L(s) = [L, A_L(s)].$$

If  $L = p_1 P_1^\delta + p_2 P_2^\delta + p_3 P_3^\delta$  and  $A = a_1(s)P_1^\delta + a_2(s)P_2^\delta + a_3(s)P_3^\delta$ , it follows that

$$\begin{cases} a'_1 = -\varepsilon(p_2 a_3 - p_3 a_2), \\ a'_2 = p_1 a_3 - p_3 a_1, \\ a'_3 = -(p_1 a_2 - p_2 a_1). \end{cases}$$

Then the Casimir function is

$$\varepsilon a_1^2 + a_2^2 + a_3^2 = k_1.$$

Proceeding as above, the orbit of  $A$  in  $E_\mu$  is lying in a certain level set  $\mathcal{C}$  of the Casimir function and we will show that is open in  $\mathcal{C}$ . Now  $\mathcal{C}$  is not always connected, however, when  $k_1 \neq 0$  the orbit agrees with one of its connected components. When  $k_1 = 0$  we can proceed as above to show that there are two connected components which are lying in the same orbit.

As a canonical element on each orbit of  $\mathfrak{o}(3, 0)$  is of the form  $aP_1^\delta$ , a canonical one in  $\mathfrak{o}(4, 0)$  is

$$aP_1^1 + bP_1^{-1}.$$

Since,  $P_1^\delta = \frac{1}{\sqrt{2}}(M_{01} + \delta M_{23})$ , the above element can also be given by

$$aM_{01} + bM_{23}.$$

On the other hand, there are three types of orbits in  $\mathfrak{o}(3, 1)$ , depending on  $k_1 > 0$ ,  $k_1 < 0$  or  $k_1 = 0$ . The canonical elements in each type of orbit are

$$\{aP_1^\delta, aP_2^\delta, (P_1^\delta + P_2^\delta)\},$$

where  $a \in \mathbb{R}$ . Therefore, we get nine classes in  $\mathfrak{o}(4, 2)$

$$\begin{aligned} &aP_1^1 + bP_1^{-1}, && aP_1^1 + bP_2^{-1}, && aP_1^1 + (P_1^{-1} + P_2^{-1}), \\ &aP_2^1 + bP_1^{-1}, && aP_2^1 + bP_2^{-1}, && aP_2^1 + (P_1^{-1} + P_2^{-1}), \\ &(P_1^1 + P_2^1) + bP_1^{-1}, && (P_1^1 + P_2^1) + bP_2^{-1}, && (P_1^1 + P_2^1) + (P_1^{-1} + P_2^{-1}) \end{aligned}$$

where  $a, b \in \mathbb{R}$ . Note that any element can be written as  $aA + bB$ , where  $A$  and  $B$  are commuting matrices. Furthermore, the simplest coordinates are obtained by taking  $A$  and  $B$  in  $\{\sqrt{2}P_1^\delta, \sqrt{2}P_2^\delta, \sqrt{2}(P_1^\delta + P_2^\delta)\}$ , better than in  $\{P_1^\delta, P_2^\delta, (P_1^\delta + P_2^\delta)\}$ .

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